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Boundary degrees of freedom in fractional quantum Hall effect: Excitations on common boundary of two samples

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Abstract

Using the Carlip's method we have derived the boundary action for the fermion Chern-Simons theory of quantum Hall effects on a planar region with a boundary. We have computed both the bulk and edge responses of currents to the external electric field. From this we obtain the well-known anomaly relation and the boundary Hall current without introducing any *ad hoc* assumptions such as the chirality condition. In addition, the edge current on the common boundary of two samples is found to be proportional to the difference between Chern-Simons coupling strengths.

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I. INTRODUCTION

The phenomenological theory of fractional quantum Hall effect (FQHE) based on the Chern-Simons gauge field coupled to interacting fermions was advanced first by Lopez and Fradkin [1], and by Halperin and coworkers [2], and has been remarkably successful in explaining the fractional Hall conductance. Recently the theory has been reformulated in aesthetically satisfying manner by Gimm and Salk [3]. This theory of FQHE consists of spinless (or completely polarized) electron field coupled to both external electromagnetic and Chern-Simons gauge fields, and is described by the action [3]

$$S=\int d^3x\,\psi^*\left(iD_0+\mu-\frac{1}{2m}\mathbf{D}^2\right)\psi+S_{CS}[a]-\frac{1}{2}\int d^3x d^3x'(|\psi(x)|^2-\bar{\rho})V(x,x')(|\psi(x')|^2-\bar{\rho}), \quad (1.1)$$

with

$$S_{CS}[a]=\int d^3x \frac{\alpha}{2}\epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho\ , \quad (1.2)$$

where $D_0=\partial_0-ie(A_0-a_0)$, $\mathbf{D}=\nabla+ie(\mathbf{A}-\mathbf{a})$, and (ψ, A_μ, a_μ) represent, respectively, the electron, external electromagnetic and Chern-Simons gauge potentials. μ and $\bar{\rho}$ denote the chemical potential and the average electron density, respectively, and V is the pair potential between electrons. If we drop the Chern-Simons action S_{CS} from (1.1), then the action describes the integral quantum Hall effect [4].

This fermion Chern-Simons theory of FQHE, however, does not deal with the possible boundary excitations of 2-dimensional quantum Hall droplet. The description of quantum Hall phenomena by Chern-Simons gauge theory was in fact initiated by the observation that the essential features of both a droplet of incompressible 2-dimensional electron gas in the quantum Hall regime and a pure Chern-Simons theory in a bounded 2-dimensional surface, are determined by the edge excitations at the boundary [5]. Wen [6] used this observation to deduce that the edge states form a representation of Kac-Moody algebra, which indicates the existence of physical observable at the boundary, and Stone [7] and Balachandran, *et al.* [8] have subsequently shown that the representation is isomorphic to that generated by a chiral

scalar field theory on the boundary. By requiring the gauge invariance at the boundary they derived the boundary action which consists of a scalar field chirally coupled to a gauge field. This has led to the (1+1)-dimensional effective field theory approach [9] that accounts for the quantum Hall effects mainly as edge phenomena.

Recently Carlip [10] had observed that the Chern-Simons action (1.2) defined on a bounded surface cannot have classical extrema due to the boundary contribution, and that the variational principle does require a modification of the action (1.2) by a boundary action. He used this observation to derive a boundary action for the Chern-Simons gravity in (2+1)-dimensions. Especially it was shown that the gauge variance of the boundary action makes the would-be gauge degrees of freedom dynamical.

The purpose of this paper is to study the boundary excitations of a quantum Hall droplet starting from the well-established fermionic Chern-Simons model of FQHE, with help of the Carlip's method. We start from the action (1.1) and use only the standard procedures of quantum field theory without introducing any *ad hoc* assumptions such as the chirality condition. We then proceed to clarify the resulting edge excitations of boundaries. Especially we wish to consider two samples joined together with a common boundary. However, the complicated contact interactions may arise if two quantum Hall samples differing in their electronic properties, are merged. To avoid such unnecessary complexity, we, instead, choose to consider a boundary formed by a stepwise external magnetic field within a sample. It turns out that this effectively generates two regions with distinct Chern-Simons coupling strengths. Subsequently, we analyze in detail the effects of the boundary to the bulk, the boundary excitations, and their experimental implications. Should the results be confirmed by experiments, it would mean that the theory (1.1) indeed provides a good field theoretic description of the quantum Hall effect. electron system.

In the next section we start from the Chern-Simons action (1.2) and show how the boundary degrees of freedom arise both by the Carlip's original method, and by the path integral approach. We then show that the so called edge phenomena can be understood from the boundary action supplemented to the bulk part in (1.2). In section III, we consider the

case of two quantum Hall droplets sharing a common boundary, study the edge excitations, and derive the boundary current. In the last section we conclude with discussions on our results.

II. BOUNDARY DEGREES OF FREEDOM IN CHERN-SIMONS GAUGE THEORY

In this section we consider the pure Chern-Simons gauge theory, and show how the gauge variance of the theory at the boundary gives rise to the physical degrees of freedom by the Carlip's original method [10] and also by the path integral approach. We then consider the theory with an external source and study the effect of these boundary degrees of freedom on the quantum Hall phenomena.

A. Carlip's method

The variation of the Chern-Simons action (1.2) defined on a three-manifold M is

$$\delta S_{CS} = \frac{\alpha}{2}\int_M d^3x \partial_\nu[\epsilon^{\mu\nu\rho} a_\mu \delta a_\rho] + \alpha \int_M d^3x \epsilon^{\mu\nu\rho} \delta a_\mu \partial_\nu a_\rho. \tag{2.1}$$

The first term of Eq.(2.1) is the boundary term which breaks the gauge invariance. This action has no classical extrema for generic variations on the boundary. In order for the theory to admit classical solutions, therefore, S_{CS} must be supplemented by a surface term so that its variation exactly cancels the first term of Eq.(2.1);

$$\delta S_{\partial M} = -\frac{\alpha}{2}\int_M d^3x \partial_\nu[\epsilon^{\mu\nu\rho} a_\mu \delta a_\rho]. \tag{2.2}$$

The exact form of this boundary action depends on the choice of boundary conditions.

For simplicity we take the spatial part of M to be a half-plane ($y < 0$) with the boundary given by $y = 0$. $x \to \theta$ and On this manifold the variation of boundary action becomes

$$\delta S_{\partial M} = -\frac{\alpha}{2}\int_{\partial M} dt dx (a_0 \delta a_1 - a_1 \delta a_0). \tag{2.3}$$

If the field component $(a_0 - a_1)$ is fixed at the boundary

$$\delta(a_0 - a_1) = 0, \tag{2.4}$$

then the boundary action is given by

$$S_{\partial M} = -\frac{\alpha}{4}\int_{\partial M} dtdx(a_0 - a_1)(a_0 + a_1). \tag{2.5}$$

Although the total action now gives meaningful classical equations of motion on the manifold $M + \partial M$, it is still not gauge invariant. One can make this gauge variance explicit by decomposing the gauge field as

$$a_\mu = \tilde{a}_\mu + \partial_\mu \Lambda, \tag{2.6}$$

where $\tilde{a}_\mu$ is a gauge fixed potential. Then the total action becomes

$$\begin{aligned} S_{M+\partial M}[a] = & \frac{\alpha}{2}\int_M d^3x \epsilon^{\mu\nu\rho}\tilde{a}_\mu \partial_\nu \tilde{a}_\rho + \frac{\alpha}{4}\int_{\partial M} dtdx(\tilde{a}_0 - \tilde{a}_1)(\tilde{a}_0 + \tilde{a}_1) \\ & + \frac{\alpha}{4}\int_{\partial M} dtdx[\partial_a \Lambda \partial^a \Lambda + 2(\tilde{a}_0 - \tilde{a}_1)(\partial_0 \Lambda + \partial_1 \Lambda)] \end{aligned} \tag{2.7}$$

where the latin index a denotes the space-time index on the boundary, *i.e.*, $a = 0, 1$, $(\partial^a = (\partial_0, -\partial_1))$ while $\mu, \nu, \rho = 0, 1, 2$. This action explicitly shows that $\Lambda(x)$ became a dynamical variable with ∂_1 legitimate kinetic term. The would-be gauge variable $\Lambda(x)$ has thus become a dynamical field on ∂M. This boundary action can be recognized as a chiral Wess-Zumino-Witten action [11].

B. Path integral approach

Although we have derived the boundary action(2.7) by requiring the existence of classical solutions on a manifold with a boundary, one might wonder why the Chern-Simons theory, which has no physical degrees of freedom, gives rise to a theory with dynamical degrees of freedom at the boundary. The origin of this dynamical degrees of freedom is the gauge non-invariance of the theory at the boundary. This fact can be better seen in the following path integral approach.

The generating functional or partition function for the Chern-Simons theory on a manifold $M + \partial M$ is defined by

$$Z = \int Da_\mu e^{i\{S_{CS}[a]+S_{\partial M}[a]\}} \tag{2.8}$$

where S_{CS} is given by Eq.(1.2) and $S_{\partial M}$ by Eq.(2.5). Following the Faddeev-Popov procedure, we introduce

$$\Delta(a) \equiv \int D\Lambda \; \delta[F(a_\Lambda) - C(x)], \tag{2.9}$$

where $\Lambda(x)$ is a gauge parameter and $F(a)$ is a gauge fixing function. We use the covariant gauge $F(a) = \partial_\mu a^\mu$ for simplicity. a_Λ in Eq.(2.9) is a gauge transformed potential $(a_\Lambda)_\mu = a_\mu - \partial_\mu \Lambda$. By using the change of variable method one can easily show that

$$\Delta(a) = [\det \frac{\delta F(a_\Lambda)}{\delta \Lambda}]^{-1} = [\det \partial_\mu \partial^\mu]^{-1} = \Delta(a_\Lambda), \tag{2.10}$$

and that $\Delta(a)$ is gauge invariant and independent of function $C(x)$.

Substituting Eq.(2.10) into Eq.(2.8) we have

$$\begin{aligned} Z &= \int Da_\mu e^{i\{S_{CS}[a]+S_{\partial M}[a]\}} \Delta^{-1}(a)\Delta(a) \\ &= [\det(\partial_\mu \partial^\mu)] \int D\Lambda Da_\mu e^{i\{S_{CS}[a]+S_{\partial M}[a]\}} \delta[F(a_\Lambda) - C]. \end{aligned} \tag{2.11}$$

By averaging over the function $C(x)$ and making gauge transformation of a_μ so that $(a_\Lambda)_\mu$ transforms back to a_μ, we obtain

$$\begin{aligned} Z &= \int DCD\Lambda Da_\mu e^{i\int C(x)^2/2\beta \; d^3x} \delta[F(a_\Lambda) - C] e^{i\{S_{CS}[a]+S_{\partial M}[a]\}} \\ &= \int DCD\Lambda Da_\mu e^{i\int C^2/2\beta \; d^3x} \delta[F(a_\Lambda) - C] e^{i\{S_{CS}[a_\mu+\partial_\mu\Lambda]+S_{\partial M}[a_\mu+\partial_\mu\Lambda]\}} \\ &= N \int D\Lambda Da_\mu e^{i\{S_{CS}[a_\mu+\partial_\mu\Lambda]+S_{\partial M}[a_\mu+\partial_\mu\Lambda]+\frac{1}{2\beta}(\partial_\mu a^\mu)^2\}} \end{aligned} \tag{2.12}$$

where β is a gauge fixing parameter and the use has been made of the fact that Z is independent of the function $C(x)$, and S_{CS} plus $S_{\partial M}$ is not gauge invariant. Note that the exponent in the last line of Eq.(2.12) is exactly the Carlip's action (2.7) in the covariant gauge.

If the classical action $S_{CS} + S_{\partial M}$ were gauge invariant, then Λ-integration in Eq.(2.12) would be trivial giving an infinite normalization constant, and the above procedure would be an ordinary way of fixing the gauge. But for the Chern-Simons theory on a manifold with a boundary, the classical action is not gauge invariant at the boundary, and the $\Lambda(x)$-integration becomes non-trivial. In fact Λ-variable becomes a dynamical variable with a bona fide kinetic term as can be seen from Eq.(2.7). We thus see how the would-be gauge variable becomes physical in this case.

C. One parameter family of the boundary actions

As pointed out by Carlip [10] the form of the boundary action depends on the choice of the boundary conditions we impose at the boundary. In deriving the boundary action (2.7) we have fixed the potential $(a_0 - a_1)$ at the boundary by requiring Eq.(2.4). If we fix $(a_0 - va_1)$ instead,

$$\delta(a_0 - va_1) = 0, \tag{2.13}$$

at the boundary with a parameter v, then we have

$$\delta S_{CS} = \frac{\alpha}{4v} \int_{\partial M} dtdx (a_0 - va_1)(a_0 + va_1), \tag{2.14}$$

and the boundary action becomes

$$S_{\partial M} = \frac{\alpha}{4v} \int_{\partial M} dtdx [\partial_a \Lambda \partial^a \Lambda + 2(\tilde{a}_0 - v\tilde{a}_1)(\partial_0 \Lambda + v\partial_1 \Lambda)]. \tag{2.15}$$

We thus see that the boundary action is dependent on the parameter v which has a dimension of velocity. We will discuss more on the role and significance of this parameter later.

D. The effect of boundary action on the quantum Hall phenomena

Although we have the boundary action for the Λ-field in Eq.(2.7), we do not yet know what the Λ-field represents, namely, whether it describes dynamics of a physical particle, a

quasiparticle, or it only describes some intermediate stage of a physical process. We therefore integrate out the Λ-field in the path integral method, and find out what the effect of the Λ-field on the quantum Hall current is.

To do this we couple the theory to an external current $J^\mu(x)$, so that the total action S_T reads

$$S_T = \int_M d^3x \left(\frac{\alpha}{2} \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho - a_\mu J^\mu \right) + \frac{\alpha}{4v} \int_{\partial M} dt dx (a_0 - v a_1)(a_0 + v a_1)$$
$$+ \frac{\alpha}{4v} \int_{\partial M} dt dx [-\Lambda(\partial_0^2 - v^2 \partial_1^2 - i\epsilon)\Lambda + 2(a_0 - v a_1)(\partial_0 \Lambda + v \partial_1 \Lambda)], \tag{2.16}$$

where we have introduced the constant v characterizing the choice of the boundary conditions, and dropped the tilde over a_μ for notational simplicity, but a_μ is understood to be the gauge fixed potential. Note that we have introduced $i\epsilon$ prescription into the kinetic term of Λ-field so that the functional integral over Λ exists. This defines the Green's function appearing in the effective action to be Feynman Green's function, and we will discuss the physical implications of this prescription later. Since the action (2.16) is quadratic in Λ-field and we have introduced the Feynman prescription, the path integral over Λ can easily be performed;

$$Z = \int D a_\mu D\Lambda e^{i S_T[a,\Lambda]}$$
$$= N \int D a_\mu e^{i S_{eff}[a]}, \tag{2.17}$$

where

$$S_{eff}[a] = \int_M d^3x [\frac{\alpha}{2} \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho - a_\mu J^\mu] + \frac{\alpha}{4v} \int_{\partial M} dt dx (a_0 - v a_1)(a_0 + v a_1)$$
$$+ \frac{\alpha}{4v} \int_{\partial M} dt dx (\partial_0 + v \partial_1)(a_0 - v a_1) \frac{1}{\partial_0^2 - v^2 \partial_1^2 - i\epsilon} (\partial_0 + v \partial_1)(a_0 - v a_1). \tag{2.18}$$

By varying $S_{eff}[a]$ with respect to the variation of a_μ and setting $\delta_a S_{eff}[a] = 0$, we obtain

$$J^\mu(x) = \alpha \theta(-y) \epsilon^{\mu\nu\rho} \partial_\nu a_\rho + \frac{\alpha}{2v} \delta(y) m^\mu \tag{2.19}$$

where $m^\mu = (m, -vm, 0)$, $\theta(y)$ is the step function, and

$$
\begin{aligned}
m &= (a_0 + va_1) - (\partial_0 + v\partial_1)\frac{1}{\partial_0^2 - v^2\partial_1^2 - i\epsilon}(\partial_0 + v\partial_1)(a_0 - va_1) \\
&= \frac{1}{\partial_0^2 - v^2\partial_1^2 - i\epsilon}(\partial_0 + v\partial_1)2v\epsilon^{ab}\partial_a a_b \\
&= 2v\frac{1}{\partial_0^2 - v^2\partial_1^2 - i\epsilon}(\partial_0 + v\partial_1)E_x.
\end{aligned} \tag{2.20}
$$

In Eq.(2.20) we have used the fact that the x-component of electric field is defined by $E_x = \epsilon^{ab}\partial_a a_b$. Note that the first term of the first line of Eq.(2.20) comes from the bulk contribution, namely, the first term of Eq.(2.18).

If we were interested in the integral quantum Hall effect, we would drop the Chern-Simons action (1.2) from Eq.(1.1), integrate over the fermion field ψ up to the one-loop order, and would obtain the effective action in the form of Eq.(2.18) without external current where a_μ now represents the fluctuation of external electromagnetic field [4]. Then the Hall current would be determined by $\frac{\delta S_{eff}}{\delta a_\mu}$, which would be exactly the current $J^\mu(x)$ given in Eq.(2.19). Therefore Eq.(2.19) determines the structure of the integral Hall current on a manifold with boundary.

If we look at the current in 3-dimensional manifold $M + \partial M$, the current J^μ of Eq.(2.19) is conserved:

$$
\partial_\mu J^\mu = 0. \tag{2.21}
$$

However, if we consider what happens only at the boundary ∂M, the current is not conserved. This is the realization of the well-known Callan-Harvey anomaly cancellation mechanism [12]. The current at the boundary is

$$
J_B^a = \frac{\alpha}{2v} m^a, \tag{2.22}
$$

and the divergence of this current is

$$
\partial_a J_B^a = \alpha\epsilon^{ab}\partial_a a_b = \alpha E_x, \tag{2.23}
$$

which is the current anomaly relation discussed in detail by the authors of Refs. [7–9], and explains the integral quantum Hall effect with α being the Hall conductivity. This shows that

the integral quantum Hall effect can be described as an edge phenomenon. Note that the anomaly relation is independent of the parameter v which represents the choice of boundary conditions. As noted above a part of the anomalous edge current J^a(through m^a) comes from the bulk action, the first term of Eq.(2.16). This fact has been discussed by authors of Refs. [9,13]. It is amusing to note that in our approach this phenomenon can be understood from the first principle once we start from the phenomenological action (1.1)

III. FRACTIONAL QUANTUM HALL EFFECT ON THE COMMON BOUNDARY OF TWO SAMPLES

In the last section we have shown that the Carlip's method of finding the boundary action applied to fermion Chern-Simons theory explains the important aspects of the integral quantum Hall phenomena by the field theoretic method without further *ad hoc* assumptions. In this section we apply the same method to the fermion Chern-Simons theory of FQHE on a manifold M, which consists of two submanifolds M_1 and M_2 sharing a common boundary ∂M. The spatial parts of M_1 and M_2 consist of the lower ($y < 0$) and the upper ($y > 0$) half-planes, respectively, and the boundary is defined by the equation, $y = 0$. In M_1 and M_2, we assume that electronic properties are the same except that electrons couple to Chern-Simons gauge fields with the coupling constants α_1 and α_2, respectively.

This system is described by the action (1.1) with the Chern-Simons term replaced by

$$\begin{aligned} S_{CS}[a,\Lambda] = &\frac{1}{2}\int_{M_1+M_2} d^3x[\theta(-y)\alpha_1 + \theta(y)\alpha_2]\epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho \\ &+\frac{\alpha_1-\alpha_2}{4}\int_{\partial M} dtdx[(a_0 - va_1)(a_0+va_1) \\ &-\Lambda(\partial_0^2 - v^2\partial_1^2)\Lambda + 2(a_0 - va_1)(a_0+v\partial_1)\Lambda]. \end{aligned} \tag{3.1}$$

Note that the fermion part of the action Eq.(1.1) is not affected by the presence of the boundary ∂M.

The Hubbard-Stratonovich transformation of the quartic interaction term and the functional integration over the fermion field yield the effective action [1,3],

$$S_{eff} = -iTr\ln(iD_0 + \mu + \lambda - \frac{1}{2m}\mathbf{D}^2) + S_{CS}[a_\mu - \delta_{\mu 0}\lambda] + S[\lambda], \tag{3.2}$$

where

$$S[\lambda] = -\int_M d^3x\lambda(x)\bar{\rho} + \frac{1}{2}\int_M d^3x d^3x'\lambda(x)V^{-1}(x-x')\lambda(x'). \tag{3.3}$$

The Chern-Simons part of the action becomes, after the functional integration over Λ-field,

$$S_{CS}[a] = \frac{1}{2}\int_{M_1+M_2} d^3x[\theta(-y)\alpha_1 + \theta(y)\alpha_2]\epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho + \frac{\alpha_1-\alpha_2}{4v}\int_{\partial M} dtdx[(a_0 - va_1)(a_0 + va_1) + (\partial_0 + v\partial_1)(a_0 - va_1)\frac{1}{\partial_v^2 - i\epsilon}(\partial_0 + v\partial_1)(a_0 - va_1)], \tag{3.4}$$

where $\partial_v^2 = \partial_0^2 - v^2\partial_1^2$.

The classical configurations $\bar{a}_\mu = \langle a_\mu(x)\rangle$ and $\bar{\lambda} = \langle\lambda(x)\rangle$ are determined by

$$\left.\frac{\delta S_{eff}}{\delta a_\mu}\right|_{\bar{a},\bar{\lambda}} = 0 \quad \text{and} \quad \left.\frac{\delta S_{eff}}{\delta\lambda}\right|_{\bar{a},\bar{\lambda}} = 0. \tag{3.5}$$

Following Gimm and Salk [3] we introduce

$$A_\mu^{eff} = A_\mu - a_\mu, \tag{3.6}$$

so that

$$\mathbf{B}^{eff} = \nabla\times\mathbf{A}^{eff} = \partial_1 A^2_{eff} - \partial_2 A^1_{eff} \tag{3.7}$$

describes the residual magnetic field after screened out by the Chern-Simons magnetic field. Then Eq.(3.5) gives

$$\langle j^\mu\rangle - [\alpha_1\theta(-y) + \alpha_2\theta(y)]\epsilon^{\mu\nu\rho}[\langle\partial_\nu A_\rho\rangle - \left\langle\partial_\nu A_\rho^{eff}\right\rangle] - \frac{\alpha_1-\alpha_2}{2v}m^\mu\delta(y) = 0, \tag{3.8}$$

$$\langle j_0\rangle - \bar{\rho} + \int_M d^3x' V^{-1}(x-x')\langle\lambda(x')\rangle = 0, \tag{3.9}$$

where $m^\mu = (m, -vm, 0)$,

$$\begin{aligned} m &= 2v(\partial_0 + v\partial_1)\frac{1}{\partial_v^2 - i\epsilon}\epsilon^{ab}\partial_a\left\langle A_b - A_b^{eff}\right\rangle \\ &= 2v(\partial_0 + v\partial_1)\frac{1}{\partial_v^2 - i\epsilon}\epsilon^{ab}\partial_a\bar{a}_b, \end{aligned} \tag{3.10}$$

and j^μ is the electromagnetic current of electrons.

Since $\langle j^0 \rangle$ is the average charge density of electrons, Eq.(3.9) leads to

$$\left\langle j^0 \right\rangle = \bar{\rho}, \quad \text{and} \quad \langle \lambda(x) \rangle = 0. \tag{3.11}$$

Then Eq.(3.8) implies, in the bulk regions M_1 and M_2,

$$B_{eff} = B(y) - \langle b(y) \rangle = B(y) - \left(\frac{\theta(-y)}{\alpha_1} + \frac{\theta(y)}{\alpha_2} \right) \bar{\rho}, \tag{3.12}$$

where B and b denote the external and Chern-Simons magnetic fields, respectively, and the coupling constants α_i are given by

$$\frac{1}{\alpha_i} = 2m_i\phi_0 \ , \ (i = 1, 2), \tag{3.13}$$

where $2m_i$'s are the numbers of Chern-Simons flux quantum ϕ_0 in each manifold M_i.

One possible setting to study the boundary effect of FQHE would be to join two different materials with different electronic properties with a common boundary. Then the average electron density $\bar{\rho}$ would be in general a function of the coordinate y. But this makes computations extremely complicated. Thus, for computational simplicity, we will use the same material, but apply external magnetic fields differently in M_1 and M_2 so that the Chern-Simons coupling constants become different. This makes the study of boundary effects very simple. This is the reason why we have formulated the theory in such a way that the average electron density is uniform over the whole manifold M as indicated by Eq.(3.11).

We thus apply an external magnetic field such that

$$B(y) = B^0_{eff} + \left(\frac{\theta(-y)}{\alpha_1} + \frac{\theta(y)}{\alpha_2} \right) \bar{\rho}. \tag{3.14}$$

Then from Eq.(3.12) we find that the effective magnetic field felt by the electron is uniform over M:

$$B_{eff} = B^0_{eff}. \tag{3.15}$$

We can then use the results of Lopez and Fradkin [1] for computation of the effective action as a function of the external field A_μ. From Eq.(3.14) we find the fractional filling factor,

$$\nu(y) = \theta(-y)\frac{1}{2m_1 p + 1} + \theta(y)\frac{1}{2m_2 p + 1}, \tag{3.16}$$

where $p = N_e/N_\phi^{eff}$ is the effective filling fraction with N_e and N_ϕ^{eff} denoting the total number of electrons and the number of effective(residual) flux quanta, respectively. The effective Landau gap is given by

$$\hbar\omega_c^{eff} = \frac{\hbar\omega_c}{1 + [\theta(-y)2m_1 + \theta(y)2m_2]p}. \tag{3.17}$$

To obtain the linear response functions we again follow Ref. [3], and separate the effective action Eq.(3.2) as

$$S_{eff} = S_{eff}^{cl} + S^{(2)}, \tag{3.18}$$

where S_{eff}^{cl} is the effective action evaluated at the classical configurations of the fields discussed above, and $S^{(2)}$ represents the fluctuations around the classical solutions. Up to the quadratic part in the fluctuation fields we have

$$\begin{aligned} S^{(2)} &= \frac{1}{2}\int_M d^3x d^3x' \delta A_\mu^{eff}(x)\Pi_{\mu\nu}(x,x')\delta A_\nu^{eff}(x') \\ &\quad + S_{CS}[\delta A_\mu - \delta A_\mu^{eff} - \delta\lambda\delta_{\mu 0}] + \frac{1}{2}\int_M d^3x d^3x' \delta\lambda(x) V^{-1}(x-x')\delta\lambda(x') \end{aligned} \tag{3.19}$$

where S_{CS} is given by Eq.(3.4), and $\Pi_{\mu\nu}$ is the polarization tensor [1].

After functional integration over $\delta\lambda(x)$, the fluctuation part of the effective action becomes, up to the first order in α_i's,

$$\begin{aligned} S_{eff}^{(2)} &= \frac{1}{2}\int_M d^3x d^3x' \delta A_\mu^{eff}\Pi_{\mu\nu}\delta A_\nu^{eff} + S_{CS}[\delta A_\mu - \delta A_\mu^{eff}] \\ &\quad + \frac{1}{2}\int_M d^3x \frac{1}{\beta}(\partial_\mu \delta A_{eff}^\mu)^2 + O(\alpha^2) \\ &= \frac{1}{2}\int_M d^3x d^3x' (\delta A_\mu - \delta a_\mu)[\Pi^{\mu\nu} - \frac{1}{\beta}\partial^\mu\partial^\nu](\delta A_\nu - \delta a_\nu) + S_{CS}[a_\mu] + O(\alpha^2) \end{aligned} \tag{3.20}$$

where we have introduced the gauge fixing condition into the effective action. Since we have evaluated $S_{eff}^{(2)}$ up to the first order in α_i's, this result is valid when the coupling constants α_i's are small or when the numbers of attached flux quanta $2m_i$ are very large.

We now have to evaluate the functional integral over δa_μ to obtain the linear response functions. To do this we write Eq.(3.20) as

$$S_{eff}^{(2)} = \frac{1}{2}\int_M d^3x d^3x' (\delta A_\mu - \delta a_\mu) M^{\mu\nu}(x,x')(\delta A_\mu - \delta a_\mu) + \frac{1}{2}\int d^3x d^3x' \delta a_\mu N^{\mu\nu}(x,x')\delta a_\nu + O(\alpha^2), \tag{3.21}$$

where $M^{\mu\nu} = \Pi^{\mu\nu} - \frac{1}{\beta}\partial^\mu\partial^\nu$ and $N^{\mu\nu}$ is defined such that the second term of Eq.(3.21) is the same as Eq.(3.4). It is convenient to write Eq.(3.21) in momentum space as

$$\begin{aligned} S_{eff}^{(2)} = & \frac{1}{2}\int d^3q\ \delta A_\mu(q) M^{\mu\nu}(q)\delta A_\nu(q) \\ & + \frac{1}{2}\int d^3q\ [\delta A_\mu(q) M^{\mu\nu}(q)\delta a_\nu(q) + \delta a_\mu(q) M^{\mu\nu}(q)\delta A_\nu(q)] \\ & + \frac{1}{2}\int d^3q\ d^3s\ \delta a_\mu(q)\tilde{M}^{\mu\nu}(q,s)\delta a_\nu(s) + O(\alpha^2), \end{aligned} \tag{3.22}$$

where

$$\tilde{M}^{\mu\nu}(q,s) = M^{\mu\nu}\delta^{(3)}(q-s) + N^{\mu\nu}(q,s). \tag{3.23}$$

Then the functional integration over δa_μ yields

$$S_{eff}^{EM}[\delta A] = \frac{1}{2}\int d^3q\ d^3s\ \delta A_\mu(q) K^{\mu\nu}(q,s)\delta A_\nu(s) \tag{3.24}$$

where

$$K^{\mu\nu}(q,s) = M^{\mu\nu}(q)\delta^{(3)}(q-s) - M^{\mu\alpha}(q)\tilde{M}^{-1}_{\alpha\beta}(q,s)M^{\beta\nu}(s). \tag{3.25}$$

Substituting Eq.(3.23) into(3.25), we find

$$K^{\mu\nu}(q,s) = N^{\mu\nu}(q,s) + O(\alpha^2), \tag{3.26}$$

and finally

$$\begin{aligned} S_{eff}^{EM}[\delta A] &= \frac{1}{2}\int_M d^3x d^3x' \delta A_\mu(x) N^{\mu\nu}(x,x')\delta A_\nu(x') + O(\alpha^2) \\ &= S_{CS}[\delta A_\mu] + O(\alpha^2) \\ &= \frac{1}{2}\int_{M_1+M_2} d^3x [\theta(-y)\alpha_1 + \theta(y)\alpha_2]\epsilon^{\mu\nu\rho}\delta A_\mu \partial_\nu \delta A_\rho \\ &\quad + \frac{\alpha_1-\alpha_2}{4v}\int_{\partial M} dt dx [(\delta A_0 - v\delta A_1)(\delta A_0 + v\delta A_1) \\ &\quad + (\partial_0 + v\partial_1)(\delta A_0 - v\delta A_1)\frac{1}{\partial_v^2 - i\epsilon}(\partial_0 + v\partial_1)(\delta A_0 - v\delta A_1)] + O(\alpha^2). \end{aligned} \tag{3.27}$$

It is instructive to note that, up to the first order in the coupling constants α_i's, the FQHE in a manifold $M = M_1 + M_2 + \partial M$ is determined by the effective Chern-Simons action like the integral quantum Hall effect.

The Hall current is determined by

$$\langle \delta j^\mu \rangle = \frac{\delta S_{eff}^{EM}}{\delta A_\mu}, \tag{3.28}$$

from which we find

$$\begin{aligned}
\left\langle \delta j^0 \right\rangle &= \delta(y) \frac{v e^2}{h} \left[\frac{1}{2m_1} - \frac{1}{2m_2} \right] \frac{\partial_0 + v\partial_1}{\partial_v^2 - i\epsilon} \delta E_x + O(\alpha^2) \\
\langle \delta j_x \rangle &= -\frac{e^2}{h} \left[\frac{1}{2m_1} \theta(-y) + \frac{1}{2m_2} \theta(y) \right] \delta E_y \\
&\quad + \frac{v e^2}{h} \delta(y) \left[\frac{1}{2m_1} - \frac{1}{2m_2} \right] \frac{\partial_0 + v\partial_1}{\partial_v^2 - i\epsilon} \delta E_x + O(\alpha^2) \\
\langle \delta j_y \rangle &= -\frac{e^2}{h} \left[\frac{1}{2m_1} \theta(-y) + \frac{1}{2m_2} \theta(y) \right] \delta E_x + O(\alpha^2),
\end{aligned} \tag{3.29}$$

where h is the Planck constant. ¿From these results one can easily show that the total current in the whole manifold M is conserved:

$$\partial_\mu \langle \delta j^\mu \rangle = 0, \tag{3.30}$$

but the current at the boundary is not:

$$\partial_a j^a = \frac{e^2}{h} [\frac{1}{2m_1} - \frac{1}{2m_2}] \delta E_x + O(\alpha^2). \tag{3.31}$$

To compute the explicit expression for the currents we need to compute the Green's function appearing in Eq.(3.29),

$$\frac{\partial_0 + v\partial_1}{\partial_0^2 - v^2 \partial_1^2 - i\epsilon} f(x) = \int_{\partial M} dt dx G(x - x') f(x'), \tag{3.32}$$

which, via Fourier transformation, can be shown to be

$$G(x) = \frac{1}{2\pi i |v|} \frac{P}{t - x/v} + \frac{1}{2|v|} \delta(t + x/v) \epsilon(t), \tag{3.33}$$

where P denotes the principal value prescription, and $\epsilon(t)$ is the sign function. For a constant external electric field δE_x, we find

$$\frac{\partial_0 + v\partial_1}{\partial_v^2 - i\epsilon}\delta E_x = -\frac{x}{v}\delta E_x. \tag{3.34}$$

Using Eq.(3.34) we finally obtain the currents,

$$\left\langle \delta j^0 \right\rangle = -\delta(y)\frac{e^2}{h}[\frac{1}{2m_1} - \frac{1}{2m_2}]x\delta E_x + O(\alpha^2) \tag{3.35}$$

$$\langle \delta j^x \rangle = -\frac{e^2}{h}[\frac{1}{2m_1}\theta(-y) - \frac{1}{2m_2}\theta(y)]\delta E_y + \delta(y)\frac{e^2}{h}[\frac{1}{2m_1} - \frac{1}{2m_2}]x\delta E_x + O(\alpha^2) \tag{3.36}$$

$$\langle \delta j^y \rangle = -\frac{e^2}{h}[\frac{1}{2m_1}\theta(-y) - \frac{1}{2m_2}\theta(y)]\delta E_x + O(\alpha^2). \tag{3.37}$$

Eq.(3.37) shows that $\langle \delta j_y \rangle$, the current in the y-direction, is discontinuous at the boundary, $y = 0$. Therefore there is a current injection to the boundary, which is responsible for the boundary current in $\langle \delta j^0 \rangle$ and $\langle \delta j_x \rangle$. Note that the boundary current is proportional to the variable x. The reason for this is that the influx of the charge coming from the discontinuity of $\langle \delta j_y \rangle$ is uniform along the boundary, and therefore the current contribution of this influx will be proportional to the length of the boundary that receives this influx of charge. This is of course another realization of the Callan-Harvey anomaly cancellation mechanism.

We note that the currents (3.35)$-$(3.37) are independent of the parameter v, which characterizes the choice of boundary conditions at ∂M. The reason for this can be traced back to the Feynman Green's function (3.33), which is dictated by the requirement of the existence of functional integral over Λ-field at the boundary.

When we use quantum field theory to describe the condensed matter system as we have been doing in this paper, what it describes is the situation when the system has reached the steady state. Thus our result implies that, after the system has reached the steady state, the Hall currents at the boundary are v-independent, and given by Eqs.(3.35)-(3.37).

However, right after the external electromagnetic field is applied to the quantum Hall sample, the system is not in the steady state, and the Green's function in Eq.(3.29) need not be the Feynman Green's function. In this situation the boundary condition for the Green's function must be supplied in accordance with the physical situation of the system, and the Hall current will be dependent on the parameter v.

IV. CONCLUSION

We have started from the fermion Chern-Simons theory of integral and fractional quantum Hall effects defined on a manifold with boundary, have used only the standard procedures of quantum field theory without introducing any new assumptions, and have derived the boundary actions and the Hall currents at the boundary. This method shows how the anomaly at the boundary arises and also the fact that a part of the anomaly comes from the boundary action (the so-called consistent anomaly) and the other from the bulk action. The structure of Hall current shows that the discontinuity of the Hall currents across the boundary is responsible for the existence of the anomaly. In our approach the anomaly arises naturally after the functional integral over Λ-field. This is contrasted to the approach of Ref. [8] where the chirality constraint $((\partial_0 - \partial_1)\Lambda = 0)$ was imposed by hand.

On the common boundary of two fractional quantum Hall samples the edge current along the boundary is given by

$$\langle \delta j_x \rangle = \frac{e^2}{h} \left(\frac{1}{2m_1} - \frac{1}{2m_2} \right) x \delta E_x, \tag{4.1}$$

where δE_x is the external electric field. This result is valid in the small α_i regime and after the system has reached the steady state. During the transient period right after the external electromagnetic field is turned on, the current will be dependent on the specific boundary conditions characterized by the parameter v. It would be interesting to find out the v-dependence of the edge current experimentally.

Our method is based on the Carlip's method where, due to the gauge variance of Chern-Simons theory at the boundary, the would-be gauge degrees of freedom become dynamical at the boundary. This dynamical degree of freedom is described by the Λ-field. Since we do not yet know what the Λ-field does physically, we have integrated it out and studied the effects through the boundary Hall current, Eq. (4.1). It would be of great interest if one could clarify its physical implications, for example, by finding a way to excite it by some means.

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REFERENCES

[1] A. Lopez and E. Fradkin, Phys. Rev. **B44**, 5246(1991), *ibid.* **B47**, 7080 (1993).

[2] B. I. Halperin, P. A. Lee, and N. Read, Phys. Rev. **B47**, 7312 (1993).
S. H. Simon, and B. I. Halperin, *ibid.* **B48**, 7368 (1993) **B50**, 1908 (1994).
S. He, S. H. Simon, and B. I. Halperin, *ibid.* **B50**, 1823 (1994).

[3] T. -H. Gimm, and S. -H. S. Salk, Phys. Rev. **B54**, 4468 (1996).

[4] E. Fradkin, *Field Theories of Condensed Matter Systems*, (Addison-Wesley, New York, 1991).

[5] E. Witten, Comm. Math. Phys. **121**, 351 (1989).
G. Moore, and N. Seiberg, Phys. Lett. **B220**, 422 (1990).

[6] X. G. Wen, NSF-ITP-89-157, ITP Santa Barbara Preprint.
X. G. Wen, Phys. Rev. Lett. **64**, 2206 (1990).

[7] M. Stone, Ann. Phys.(N.Y.) **207**, 351 (1991).

[8] A. P. Balachandran, G. Bimonte, K. S. Gupta and A. Stern, Int. J .Mod .Phys. **A7**, 4655 and 5855 (1992).
A. P. Balachandran, L. Chandar, and A. Momen, Nucl. Phys. **B461**, 581 (1996).

[9] L. Chandar, Mod. Phys. Lett. **A9**, 3403 (1994).
Y. -C. Kao, and D. -H. Lee, cond-mat/9608113 and references therein.

[10] S. Carlip, Phys. Rev. **D51**, 632 (1995); J. Kor. Phys. Soc. **28**, S447 (1995); Talk given at Pacific Conference on Gravitation and Cosmology, Seoul (gr-gc/9603049) and references therein.

[11] J. Wess and B. Zumino, Phys. Lett. **B32**, 95 (1971).
E. Witten, Nucl. Phys. **B223**, 422 (1983).

[12] C. G. Callan, Jr. and J. A. Harvey, Nucl. Phys. **B250**, 427 (1985).

[13] S. G. Naculich, Nucl. Phys. **B296**, 837 (1988); S. Chandrasekharan, Phys. Rev. **D49**, 1980 (1994).

A. P. Balachandran, L. Chandar, B. Sathiapalan, Nucl. Phys. **B443**, 465 (1995).

N. Maeda, Phys. Lett. **B376**, 142 (1996).